Uniform Circular Motion

A particle is in uniform circular motion if it travels in a circular path at a constant speed. Orbiting satellites, ceiling fans and vinyl records are a few objects that display uniform circular motion. Even though its speed remains constant, a particle in uniform circular motion is constantly accelerating. This can initially be a little counterintuitive, but it stems from the definition that acceleration is the rate of change of velocity which is a vector quantity, having both magnitude and direction. Since the direction of the speed continuously changes when a particle moves in a circular path, the velocity also continuously changes and hence there is a constant acceleration.

Centripetal Acceleration

Consider a particle moving in a circular path of radius r with a constant speed v, as shown in Fig. 1. At a particular time t let the velocity of the particle be $\vec{v_1}$ and after a short time Δt let the velocity be $\vec{v_2}$. Here, $\Delta\theta$ is the angle swept and s is the distance traveled by the particle.

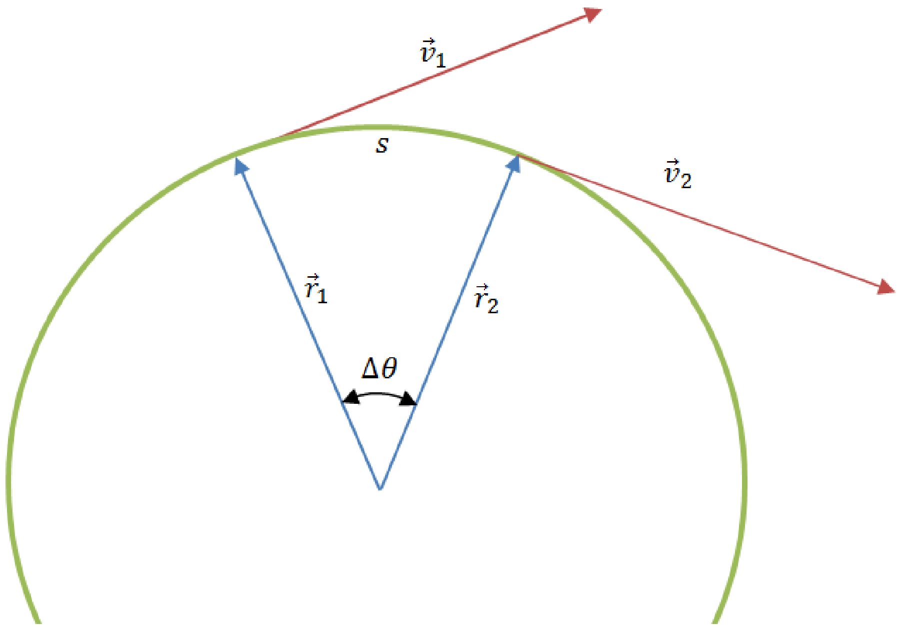


Fig. 1: Particle in uniform circular motion

The average acceleration can then be defined as

$$\vec{a}_{avg} = \frac{\vec{v}_2 - \vec{v}_1}{\Delta t} = \frac{\Delta \vec{v}}{\Delta t}$$

... Eq. (1)

The following image shows the subtraction of the of the two velocity vectors.

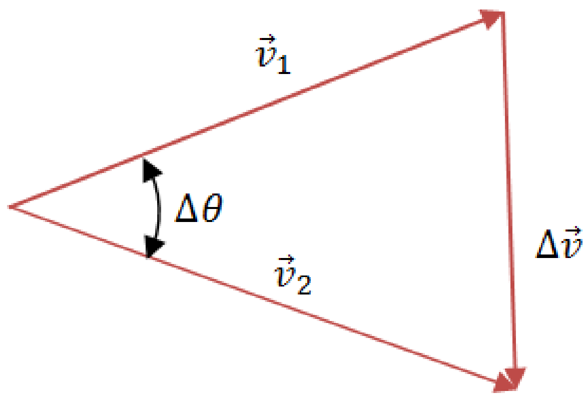


Fig. 2: Vector subtraction

If the time interval is shortened then $\Delta \vec{v}$ becomes smaller and almost perpendicular to $\vec{v}_1$, as shown below.

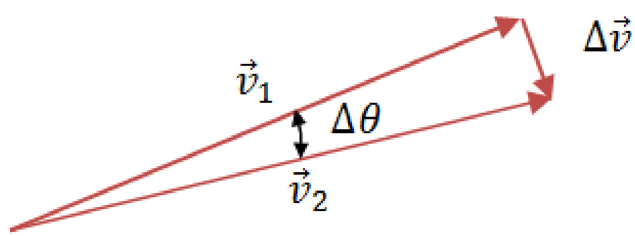


Fig. 3: Vector subtraction - small angle

For a very small angle, using geometry, we can write

$$\Delta v \approx v_1 \cdot \Delta\theta \approx v \cdot \Delta\theta$$

Also, from geometry,

$$\Delta\theta = \frac{s}{r} = \frac{v \cdot \Delta t}{r}$$

If we take the limit as $\Delta t \rightarrow 0$, we can write

$$dv = v \cdot d\theta \quad \text{and} \quad d\theta = \frac{v \cdot dt}{r}$$

Therefore, the magnitude of the acceleration, at the instant when the velocity is $\vec{v}_1$, is

$$\frac{dv}{dt} = \frac{v \cdot v \cdot dt}{r \cdot dt}$$

which simplifies to

$$a = \frac{v^2}{r}$$

... Eq. (2)

The direction of this acceleration can be obtained from Fig. 3. As $\Delta\theta \rightarrow 0$, $\Delta \vec{v}$ becomes perpendicular to $\vec{v}_1$ and points radially inward to the center of the circle. This acceleration is called the **centripetal acceleration.**

Example 1: Centripetal Acceleration

Problem Statement: A disk with a radius of 0.1m is spinning about its central axis at a uniform rate. The velocity of a point on the edge of the disk are 1 m/s. What is the centripetal acceleration of a point on this disk located 0.05m from the axis of rotation.

restart;

Data:

$v_1 := 1 :$ [m/s]

$r_1 := 0.1 :$ [m]

r_2 [m]

$:= 0.\backslash$
$05 :$

Solution:

Since the two points are located at different distances from the axis of rotation, they cover different distances per unit time. However, the angle swept per unit time stays the same. The angle swept by the outer point is

$$\theta = \frac{v_1 \cdot t}{r_1}$$

and the angle swept by the inner point is

$$\theta = \frac{v_2 \cdot t}{r_2}$$

Equating these two angles, yields

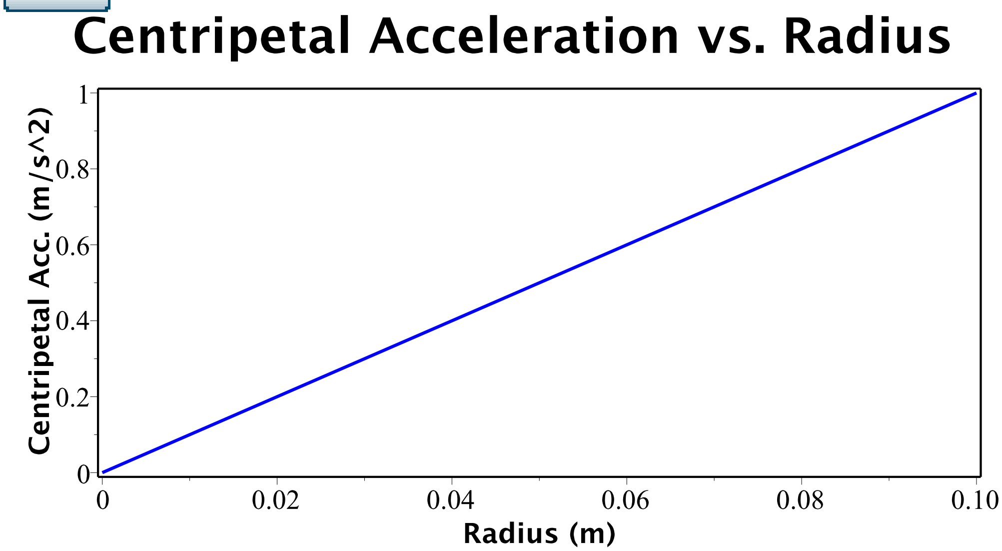
$$v_2 := \frac{v_1 \cdot r_2}{r_1} :$$

And the centripetal acceleration using Eq. (2) is,

$$a := \frac{(v_2)^2}{r_2} = 5.00$$

Therefore a point located 0.05m from the central axis experiences a centripetal acceleration of 5 m/s^2. The following is a plot of the centripetal acceleration vs. radius for points on the disk. This plot shows that the centripetal acceleration increases linearly with the radius when all the points are rotating together with the same rate of angular motion.

Plot for Centripetal Acceleration vs. Radius



Time Period and Frequency

The time period T is defined as the time taken by an object in uniform circular motion to complete one full circle.

$$T = \frac{2 \cdot \pi \cdot r}{v}$$

... Eq. (3)

Here r is the radius of the circular path and v is the speed. Additionally, the frequency f is defined as the number of full circles the object completes per unit time.

$$f = \frac{1}{T}$$

... Eq. (4)

An Example with MapleSim

Example 2: Human Centrifuge

Problem Statement: A human centrifuge, used for training fighter pilots, is rotating at a rate of 45 rpm. The distance between the pilot and the axis of rotation is 5m. What is the centripetal acceleration of the pilot?

Analytical Solution

restart :

Data:

$n := 45 :$ [rpm]

$r := 5 :$ [m]

$g := 9.\backslash 81 :$ [m/s²]

Solution:

Since the rate of rotation is 45 rpm, the speed of the pilot is

$$v := \frac{n \cdot 2 \cdot \mathrm{Pi} \cdot r}{60}$$

$$\frac{15}{2}\pi \qquad \textbf{(3.1.1.1)}$$

The magnitude of the centripetal acceleration, given by Eq. (2) is

$$a := \frac{v^2}{r}$$

$$\frac{45}{4}\pi^2 \qquad \textbf{(3.1.1.2)}$$

$$\xrightarrow{\text{at 5 digits}}$$

$$111.03 \qquad \textbf{(3.1.1.3)}$$

In *g* units, this is

$$a_g := \frac{a}{g}$$

$$\frac{45}{4}\frac{\pi^2}{g} \qquad \textbf{(3.1.1.4)}$$

$\xrightarrow{\text{at 5 digits}}$

$$\frac{111.03}{g} \qquad \textbf{(3.1.1.5)}$$

Therefore the pilot experiences approximately 11g of centripetal acceleration.

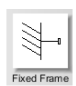
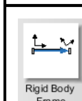
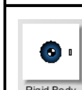
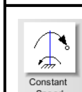
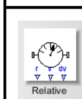
MapleSim Simulation

Constructing the Model

Step 1: Insert components

Drag the following components into the workspace:

Table 1: Components and locations

Component	Location
	Multibody > Body and Frames
	Multibody > Joints and Motions
	Multibody > Body and Frames
	Multibody > Body and Frames
	1-D Mechanical > Rotational > Motion Drivers
	1-D Mechanical > Rotational > Motion Drivers

Step 2: Connect the components

Connect the components as shown in the following diagram (the dashed boxes are not part of the model, they have been drawn on top to help make it clear what the different components are for).

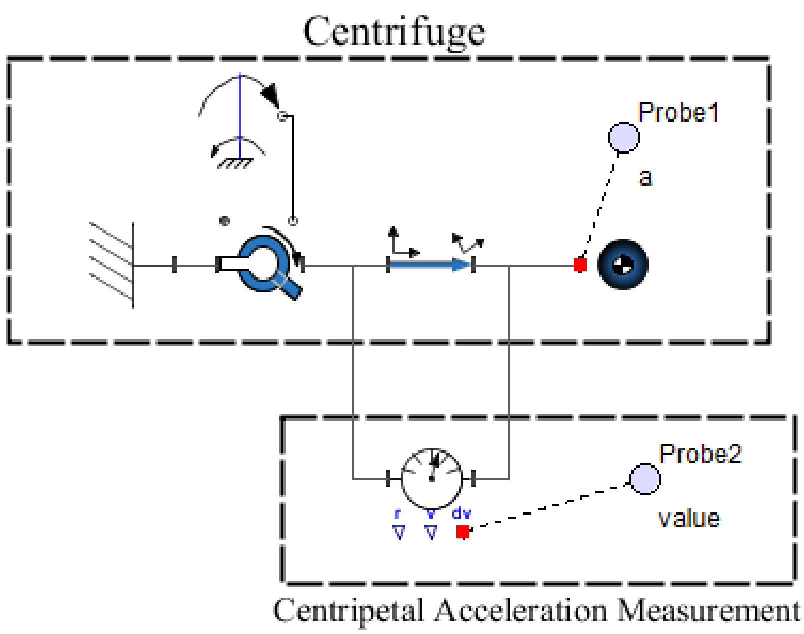


Fig. 4: Component diagram

Step 3: Create parameters

Click the **Add a Parameter Block** icon ( ), click on the workspace and double click the **Parameters** icon that appears on the workspace. Create parameters for the rate of rotation n and the distance from the axis of rotation r (as shown below).

Parameters subsystem default settings

Name	Type	Default Value	Default Units	Description
n	Real	1		Rate of rotation (rpm)
r	Real	1		Distance from axis of rotation (m)

Fig. 5: Parameter Block

Step 4: Adjust the parameters

Return to the main diagram (> Main ▶) and, with a single click on the **Parameters** icon, enter the following parameters (Fig. 6) in the **Inspector** tab.

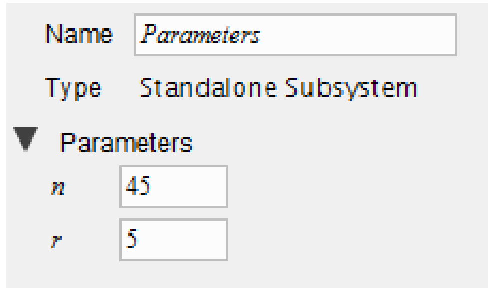


Fig. 6: Parameters

Note: Step 3 and Step 4 are not essential and can be skipped. The parameter values can be directly entered for each component instead of using variables. However, creating a parameter block as described above makes it easy to repeatedly change the parameters and play around with the model to see the effects on the simulation result.

Step 5: Set up the Centrifuge

1. Click the **Revolute** component and, in the **Inspector** tab, select **[0,1,0]** for the axis of rotation ($\widehat{e}_1$).
2. Click the **Rigid Body Frame** component and enter **[r,0,0]** for the x,y,z offset ($\overline{r}_{XYZ}$).
3. Click the **Rotational Constant Speed** component and enter ***n*** for the fixed speed ($\boldsymbol{\omega}_{fixed}$). Also, change the units to rpm.
4. Click the **Probe** connected to the **Rigid Body** component and select **1**, **2**, and **3** under **Acceleration** in the **Inspector** tab. This probe will show the acceleration components along the x, y and z axes.

Step 6: Set up the sensor for the centripetal acceleration

1. Connect the **Relative Translation** sensor across the **Rigid Body Frame** component in a parallel configuration, as shown in Fig. 4.
2. Attach a **Probe** to the **dv** port of the **Relative Translation** sensor.

3. Click this **Probe** and select **1** in the **Inspector** tab. This probe will show the sensor measurement of the acceleration in the direction along the frame that connects the object to the axis of rotation. This is the centripetal acceleration.

Step 7: Run the Simulation

In the **Settings** tab, reduce the **Simulation duration (t_d)** to 1 s (a long simulation time is not required). Then run the simulation.

The following image shows the 3-D view for the simulation.

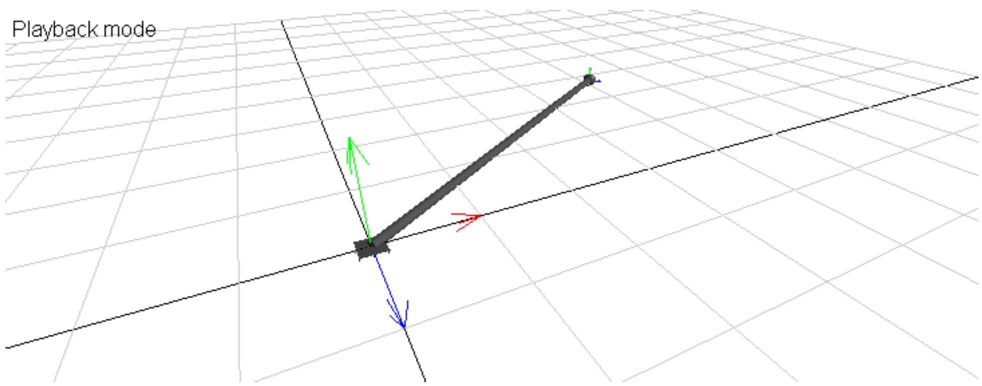


Fig. 7: A 3-D view of the Centrifuge model

Reference:
Halliday et al. "Fundamentals of Physics", 7th Edition. 111 River Street, NJ, 2005, John Wiley & Sons, Inc.